Tuesday, February 24, 2015 4:37 PM

Okay, so w/ ∇_μ we can take derivatives. What else? Well secretly ∇_μ is also telling us how to move vectors around in a space (from tangent space to tangent space!)

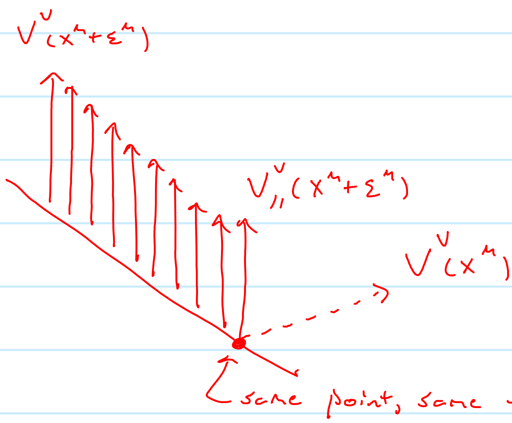
Consider: $\partial_\mu V^\nu \Big|_{x^\mu} = \lim_{\varepsilon^\mu \to 0} \dfrac{V^\nu(x^\mu+\varepsilon^\mu) - V^\nu(x^\mu)}{\varepsilon^\mu}$ ← but this difference can't make sense since the V^ν's live in different spaces

$$\Downarrow$$

$\nabla_\mu V^\nu \Big|_{x^\mu} = \lim_{\varepsilon^\mu \to 0} \dfrac{V^\nu_{\parallel}(x^\mu+\varepsilon^\mu) - V^\nu(x^\mu)}{\varepsilon^\mu}$

$V^\nu_{\parallel}(x^\mu+\varepsilon^\mu)$ means we take the vectors value at $x^\mu+\varepsilon^\mu$ and we //-transport it back to x^μ before subtracting.

//-transport (Keep the vector // to itself)



We can get a formal definition of //-transport using ∇_μ:

Suppose we want to //-transport V^ν along a curve $x^\mu(\lambda)$

Recall: $\dfrac{d}{d\lambda} = \dfrac{dx^\mu}{d\lambda}\partial_\mu$

Which we should now replace w/: $\dfrac{D}{d\lambda} = \dfrac{dx^\mu}{d\lambda}\nabla_\mu = \dfrac{d}{d\lambda} + \Gamma^{\circ}_{\mu\circ}\dfrac{dx^\mu}{d\lambda}$ ← $\partial_\mu + \Gamma^{\circ}_{\mu\circ}$

Then to //-transport V^ν along $x^\mu(\lambda)$ we insist $\dfrac{DV^\nu}{d\lambda} = \dfrac{dx^\mu}{d\lambda}\nabla_\mu V^\nu = 0$

or $\boxed{\dfrac{dV^\nu}{d\lambda} + \Gamma^\nu_{\mu\rho}\dfrac{dx^\mu}{d\lambda}V^\rho = 0}$

//-transport of vectors

For tensors: $\frac{D}{d\lambda} T^{\alpha}{}_{\beta} = \frac{dx^{\mu}}{d\lambda} \nabla_{\mu} T^{\alpha}{}_{\beta} = \frac{dx^{\mu}}{d\lambda} \left(\partial_{\mu} T^{\alpha}{}_{\beta} + \Gamma^{\alpha}_{\mu\eta} T^{\eta}{}_{\beta} - \Gamma^{\delta}_{\mu\beta} T^{\alpha}{}_{\delta} \right) = 0$

Given a path $x^{\mu}(\lambda)$ and the value of $T^{\alpha}{}_{\beta}$ at $x^{\mu}(\lambda_0)$ we can solve this diff-EQ for $T_{\parallel}{}^{\alpha}{}_{\beta}$ at any other point.

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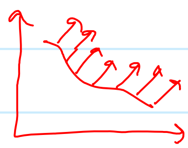
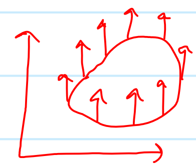
//-transport is important because:

a) It is part of constructing a consistent derivative ∇_μ

b) It helps detect curvature: $\mathbb{R}^2$

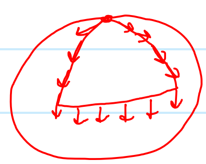


or $\mathbb{R}^2$



around loop we get no change

contrast w/ S^2



From above:

$U^\nu(\lambda_i)$ $U^\nu(\lambda_f)$

c) It helps us identify geodesic paths which encode the response of particles to the curvature of space.

So far we have been working w/ ∂ derivatives.

Covariant Derivative: $\nabla_\mu V^\nu = \lim_{\Delta x^\mu \to 0} \dfrac{V_{\parallel}^{\nu}(x^\mu + \Delta x^\mu) - V^\nu(x^\mu)}{\Delta x^\mu}$

Directional Covariant Derivative: $\dfrac{DV^\nu}{d\lambda} = \lim_{\Delta\lambda \to 0} \dfrac{V_{\parallel}^{\nu}(x^\mu(\lambda + \Delta\lambda)) - V^\nu(x^\mu(\lambda))}{\Delta\lambda} = \dfrac{dx^\mu}{d\lambda} \nabla_\mu V^\nu$

Both use $\parallel$-transport so both are covariant. The difference is the path.

In $\nabla_\mu V^\nu$ we pick a value of μ (a coordinate) and the path is a shift along x^μ. In $\frac{DV^\nu}{d\lambda}$ we shift along the curve $x^\mu(\lambda)$ which can be arbitrary, i.e. not aligned along coordinate axes! Of course if we know how the curve changes w/ coordinates $\frac{dx^\mu}{d\lambda}$, and we know how the vector changes w/ coordinates $\nabla_\mu V^\nu$, then we can combine these to determine how the vector changes along the curve $\frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu$, i.e. $\frac{DV^\nu}{d\lambda}$.

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Geodesics

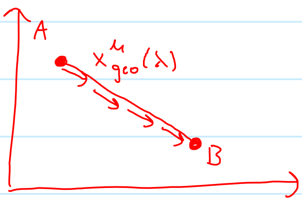
Recall: E+M (Maxwell + Lorentz Force w/ Newton) GR (EE + Geodesics)

There are 2 ways to define geodesic paths $x^\mu(\lambda)$:

1) Curves $x^\mu(\lambda)$ which extremize the distance between two points.

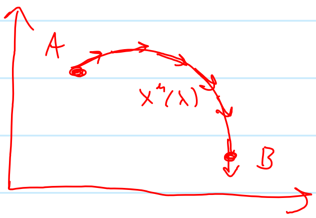
2) Curves $x^\mu(\lambda)$ which //-transport their own tangent vectors.

In $\mathbb{R}^2$:



We know that straight lines are the shortest paths and we can see that the tangent vectors are //-transported.

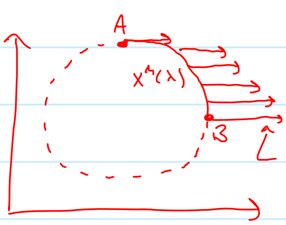
Consider a nongeodesic path in $\mathbb{R}^2$:



Note: Tangent vectors are not //!

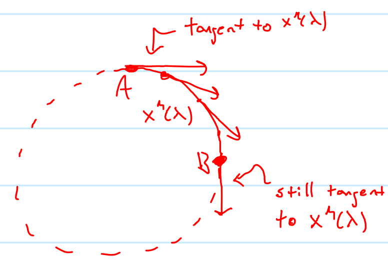
To really drive the point home consider //-transporting a vector along a circle in 2 different spaces:

$\mathbb{R}^2$:



note: What started as a tangent vector is no longer tangent!

S^1:



But notice:

Not the shortest distance between A + B!

The shortest distance between A + B!

To formalize the definition of geodesics by //-transport recall that a curve $x^\mu(\lambda)$ has components $\frac{dx^\mu}{d\lambda}$ of its tangent vector.

If $x^\mu(\lambda)$ is a geodesic (call it $x^\mu_{geo}(\lambda)$) then these components should be covariantly constant along the curve, i.e. // to each other.

Then: $x^\mu(\lambda) = x^\mu_{geo}(\lambda)$ if

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 = \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda}$$
$$= \frac{dx^\nu}{d\lambda}\left(\partial_\nu \frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\lambda}\right)$$

The geodesic equation $\boxed{0 = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\alpha}\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda}}$

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \frac{dx^\nu}{d\lambda}\partial_\nu\left(\frac{dx^\mu}{d\lambda}\right)$$

To use this first note that it is 2nd order so we need 2 boundary conditions before solving. We could give an initial position $x^\mu(\lambda_0)$ and "velocity" $\frac{dx^\mu}{d\lambda}\big|_{\lambda_0}$ and then this generates the geodesic "launched" from these.
Alternatively, and more familiar, we could give an initial and final position and this gives the extremal path between them.

Since $\Gamma^\mu_{\nu\alpha}$ depends on $g_{\mu\nu}$, the explicit form will vary for different geometries.

An intuitive example: In $\mathbb{R}^3$ w/ $(x,y,z) \Rightarrow \Gamma = 0 \Rightarrow \frac{d^2x^\mu}{d\lambda^2} = 0$ for geodesics

$$\Downarrow$$

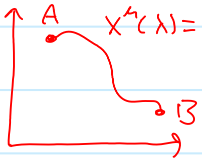
$$\underbrace{x^\mu(\lambda) = \lambda \Sigma^\mu + x^\mu_0}_{}$$

constants set by boundary conditions

A straight line! Clearly the shortest path in $\mathbb{R}^3$.

To see the extremization more explicitly consider another example.

$\mathbb{R}^2$ w/ (r,θ) $\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow \Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$



$x^\mu(\lambda) = (r(\lambda), \theta(\lambda))$

Parameterize w/ $\lambda = s$ = distance along the curve, i.e. $x^\mu(s) = (r(s), \theta(s))$

Then the total length is: $S_{AB} = \int_A^B ds = \int_A^B \sqrt{dr^2 + r^2 d\theta^2} = \int_A^B \sqrt{\frac{dr^2}{ds^2} + r^2 \frac{d\theta^2}{ds^2}}\, ds$

Extremizing this is akin to extremizing an action $S = \int L(x,v)\,dt$ in CM.

Normally: $\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0$

If we call $\frac{dr}{ds} = v_r$, $\frac{d\theta}{ds} = v_\theta$ then:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial v_r}\right) - \frac{\partial L}{\partial r} = \frac{d^2 r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial v_\theta}\right) - \frac{\partial L}{\partial \theta} = \frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0$$

Compare this to the geodesic equation $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\alpha}\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0$ w/ Γ's above and $\lambda = s$

$$\Downarrow$$

$$\frac{d^2 r}{ds^2} + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{ds}\right)^2 = \frac{d^2 r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + \Gamma^\theta_{\theta r}\frac{d\theta}{ds}\frac{dr}{ds} + \Gamma^\theta_{r\theta}\frac{dr}{ds}\frac{d\theta}{ds} = \frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0$$

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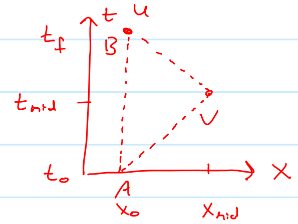
If we worked in a Lorentzian signature space, e.g. Minkowski then $\tau_{AB} = \int_A^B \sqrt{-ds^2}$ and timelike geodesics actually maximize the spacetime length!

To appreciate this consider a geodesic (constant velocity or at rest) object u in $\mathbb{M}^2$ and an accelerated non-geodesic object V.

$ds^2 = -dt^2 + dx^2$



For u: $s_{AB} = \int_{t_o}^{t_f} dt^2 = t_f - t_o$

$$V: \; s_{AB} = \int_A^B \sqrt{dt^2 - dx^2} = \int_A^B \sqrt{1-v_x^2}\, dt = \int_0^{t_{mid}} \sqrt{1-v_x^2}\, dt + \int_{t_{mid}}^{t_f} \sqrt{1-(-v_x)^2}\, dt$$
$$= \int_0^{t_f} \sqrt{1-v_x^2}\, dt < t_f - t_o$$

B.T.W. These two trajectories are representative of those in the twin paradox. The twin who remains on Earth follows u and is older than the one that travels away and then back along V.